

Finite temperature Dicke phase transition of a Bose-Einstein condensate in an optical cavity

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Dicke model predicts a quantum phase transition from normal to superradiant phases for a two-level atomic ensemble coupled with an optical cavity at zero temperature. In a recent pioneer experiment [Nature **464**, 1301 (2010)], such a phase transition has been observed using a Bose-Einstein condensate (BEC) in an optical cavity. Compared with the original Dicke model, the experimental system features finite temperature and strong atom-photon nonlinear interaction in BEC. In this Letter, we develop a finite temperature theory for the Dicke phase transition of a BEC in an optical cavity, taking into account the atom-photon nonlinear interaction. In addition to explaining the experimentally observed transition from normal to superradiant phases at finite-temperature, we point it out that a new phase, the coexistence of normal and superradiant phases, was also observed in the experiment. We show rich finite temperature phase diagrams existing in the experimental system by tuning various experimental parameters. We find that the specific heat of the BEC can serve as a powerful tool for probing various phases.

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The Dicke model, a textbook paradigm in quantum optics, describes a two-level atomic ensemble coupled with an optical cavity. It was first introduced to illustrate the importance of collective and coherent excitations for atoms induced by a single photon mode [1]. With increasing atom-cavity coupling, the Dicke model predicts a quantum phase transition from a normal phase (NP) to a superradiant phase (SP), where both the atomic ensemble and photon acquire macroscopically collective excitations [2–4]. However, due to the ‘no-go theorem’ originating from the Thomas-Reiche-Kuhn sum rule for the oscillator strength, this phase transition cannot be realized in typical cavity quantum electrodynamics [5, 6]. The experimental breakthrough for observing the quantum phase transition only occurs recently using the momentum eigenstates of a Bose-Einstein condensate (BEC) coupled with an optical cavity [7, 8]. In this pioneer experiment, there exists a nonlinear atom-photon interaction [9], which, induced by the optical lattice potential and greatly enhanced by the large atom number N , can reach the same order of the effective cavity frequency and even go beyond. In this strong interaction regime, rich dynamical properties [10, 11] as well as new quantum phase transitions [12] have been predicted at zero temperature.

Although such ideal quantum phase transitions should occur at absolute zero temperature [13] in principle, realistic experiments used to observe the quantum phase transition must be performed at finite temperature. For instance, the typical temperature of the BEC in [7, 8] for observing the Dicke phase transition is ~ 50 nK. At finite temperature, thermal fluctuations may induce new exotic phenomena beyond the prediction of the zero tempera-

ture theory [14]. A well-known example is the loss of long range superfluid order in low dimensions (two or one) at any temperature [15], where the superfluid physics is characterized by the Berezinskii-Kosterlitz-Thouless transition [16, 17]. Therefore it is crucially important to investigate the Dicke phase transition at finite temperature to fully understand the realistic experiment.

In this Letter, we present a finite-temperature field theory for the Dicke phase transition in a BEC coupled with an optical cavity to understand the recent breakthrough experiment in this system. Our main findings are the following:

(I) The experimentally observed phase diagram for the normal-superradiant phase transition, (*i.e.*, Fig. (5) in Ref. [7]) can be well understood in our finite-temperature theory. More interestingly, we show that a new phase, called CE_{SP} , can be identified in the experimental phase diagram in [7]. In the CE_{SP} phase, NP and SP coexist, but the SP is stable and the NP is metastable.

(II) By varying various physical parameters (temperature, the coupling strength, the nonlinear interaction, etc.), we show there exist rich finite-temperature phases in the experimental system, including NP, SP, CE_{SP} , CE_{NP} (coexistence of stable NP and metastable SP), as well as dynamical unstable phases (US). In certain parameter region, there is a four-phase coexistence point.

(III) We show, both analytically and numerically, that the specific heat of the BEC increases rapidly in the SP/ CE_{SP} (exponential increase at low temperature) with increasing temperature, and has a large jump at the critical transition temperature where the system becomes the NP/ CE_{NP} . Therefore the specific heat of the BEC may serve as a powerful tool for detecting different phases in

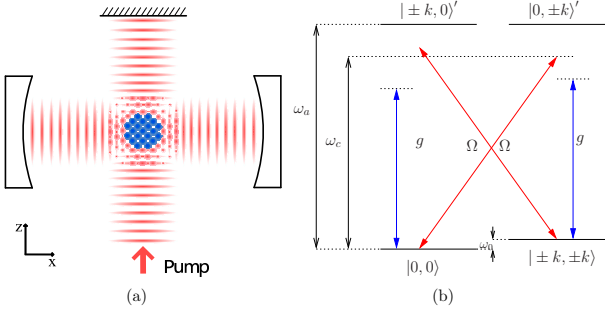


FIG. 1: (Color online) (a) A BEC with the ^{87}Rb atoms interacts strongly with a high-finesse optical cavity. (b) A four-level model is considered by introducing the zero momentum state $|0,0\rangle$ ($= |p_x, p_z\rangle$ with p_x and p_z being the momenta in the x and z directions, hereafter), the excited states $|\pm k, 0\rangle$ and $|0, \pm k\rangle$, and the symmetric superposition of states $|\pm k, \pm k\rangle$. In the dispersive limit, the excited-state levels can be eliminated adiabatically, and an effective two-level system with the zero momentum state and the symmetric superposition of states is thus formed.

the temperature-driven Dicke phase transition.

Figure 1 shows the experimental setup in which all atoms in a ^{87}Rb BEC interact identically with a single-mode photon induced by a high-finesse optical cavity [7]. In the experimental scheme, the atom momenta are used to define the spin states $|\uparrow\rangle \equiv |\pm k, \pm k\rangle$, $|\downarrow\rangle \equiv |0, 0\rangle$ with the corresponding $\text{SU}(2)$ collective spin operators $S_+ = S_-^\dagger = \sum_i |\uparrow\rangle_{ii} \langle \downarrow|$ and $S_z = \sum_i (|\uparrow\rangle_{ii} \langle \uparrow| - |\downarrow\rangle_{ii} \langle \downarrow|)$, as shown in Fig. 1(b). As a result, not only the ‘no-go theorem’ is overcome, but also the superradiant-normal phase transition condition can be satisfied [19–21], since the smaller energy scale of the effective two levels can be achieved. Under the new spin basis, the dynamics of the atom-cavity system are governed by the Hamiltonian [7]

$$H = (\omega + \frac{U}{2N} S_z) \psi^\dagger \psi + \frac{\omega_0}{2} S_z + \frac{g}{\sqrt{N}} (\psi + \psi^\dagger) (S_+ + S_-). \quad (1)$$

Henceforth we set $\hbar = 1$. ψ^\dagger denotes the photon creation operator. The effective cavity frequency $\omega = -\Delta_c + NU_0(1 + \Xi)/2$ with $\Delta_c = \omega_p - \omega_c$, where ω_c is the cavity frequency, ω_p is the pump laser frequency, $\Xi = 3/4$, and $U_0 = g_0^2/(\omega_p - \omega_a)$ with g_0 being the coupling strength between a single atom and the photon and ω_a being the atomic transition frequency. The nonlinear atom-photon interaction $U = N\Xi U_0$, which is greatly enhanced by the large atom number N , and can reach the same order as the effective cavity frequency ω or even beyond. Therefore a strong nonlinear atom-photon interaction regime is accessible in experiments. The effective atomic frequency $\omega_0 = 2\omega_r$ with the atomic recoil energy $\omega_r = k^2/2m$. The collective coupling strength $g = g_0\Omega\sqrt{N}/2(\omega_p - \omega_a)$, and Ω is the maximum pump Rabi frequency.

The finite-temperature properties of the BEC-cavity system can be obtained by calculating the partition functional of the system through the imaginary-time path-integral approach. In this procedure, we rewrite the collective spin operators in the Hamiltonian (1) as $S_z = \sum_{i=1}^N (\alpha_i^\dagger \alpha_i - \gamma_i^\dagger \gamma_i)$ and $S_+ = \sum_{i=1}^N \alpha_i^\dagger \gamma_i$ using Fermi operators $\alpha_i^\dagger(\alpha_i)$ and $\gamma_i^\dagger(\gamma_i)$, which obey the anti-commutation relation $\{\alpha_i^\dagger, \alpha_j\} = \{\gamma_i^\dagger, \gamma_j\} = \delta_{ij}$. After a straightforward calculation, the effective partition function can be written as [18]

$$\frac{Z}{Z_0} = \frac{\int [d\eta] \exp(-S)}{\int [d\eta] \exp(-S_f)}, \quad (2)$$

where $[d\eta]$ is the functional measure. In Eq. (2), the free action of the bosonic field $S_f = \int_0^\beta d\tau \sum_{i=1}^N [\alpha_i^*(\tau) \partial_\tau \alpha_i(\tau) + \gamma_i^*(\tau) \partial_\tau \gamma_i(\tau)] + \int_0^\beta d\tau \psi^*(\tau) (\partial_\tau + \omega) \psi(\tau)$, where $\tau = it$, $\partial_\tau = \partial/\partial\tau$, $\beta = 1/(k_B T)$ with k_B being the Boltzmann constant and T being the system’s temperature. The total action S is expressed, in the basis of $\Phi_i(\tau) = [\gamma_i(\tau), \alpha_i(\tau)]^T$, as

$$S = S_0(\psi, \psi^*) + \int_0^\beta d\tau \Phi_i^\dagger(\tau) G(\psi, \psi^*) \Phi_i(\tau), \quad (3)$$

where $S_0(\psi, \psi^*) = \int_0^\beta d\tau \psi^*(\tau) \partial_\tau \psi(\tau)$ and

$$G = \begin{pmatrix} \partial_\tau + \frac{\omega_0}{2} + \frac{U}{2N} \psi^* \psi & \frac{g}{\sqrt{N}} (\psi^* + \psi) \\ \frac{g}{\sqrt{N}} (\psi^* + \psi) & \partial_\tau - \frac{\omega_0}{2} - \frac{U}{2N} \psi^* \psi \end{pmatrix}. \quad (4)$$

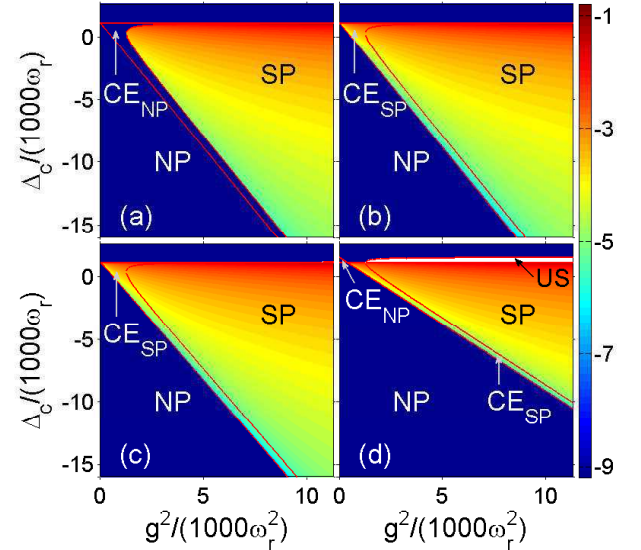


FIG. 2: (Color online) The scaled mean-photon number as the functions of the detuning Δ_c and the square of the collective coupling strength g for the temperature $T = 0.00$ nk (a), $T = 0.13$ nk (b), $T = 50.00$ nk (c), and $T = 150.00$ nk (d). The nonlinear interaction $U = 1.70 \times 10^3 \omega_r$ and the recoil frequency $\omega_r = 2\pi \times 3.71$ kHz. $1 \text{ nk} \leftrightarrow 130.00 \text{ Hz}$.

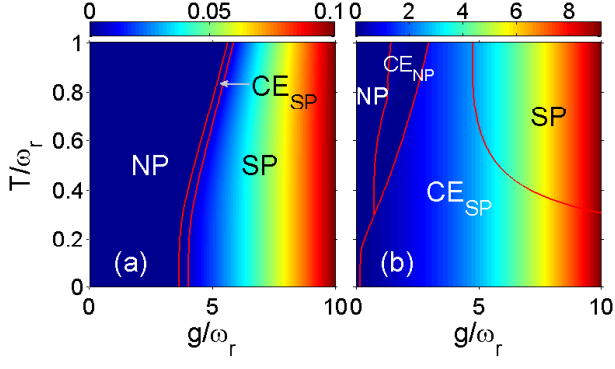


FIG. 3: (Color online) The scaled mean-photon number as the functions of the temperature T and the collective coupling strength g for $U = 10.00 \omega_r$, $\Delta_c = -20.00\omega_r$ (a) and $U = 31.10 \omega_r$, $\Delta_c = 20.00\omega_r$ with $\omega_r = 2\pi \times 3.71$ kHz.

We first integrate out the Fermi field $\Phi_i(\tau)$ (thus the atom spin degrees of freedom) in Eq. (2), and then use the standard stationary phase approximation [22], *i.e.*, $\delta S(\psi^*, \psi)/\delta\psi(\tau) = 0$ and $\delta S(\psi^*, \psi)/\delta\psi^*(\tau) = 0$, to obtain the finite temperature equilibrium phase diagram in the large atom number, which is governed by the following equation

$$\omega\psi = F(\psi^*, \psi) \tanh[\beta G(\psi^*, \psi)] \psi, \quad (5)$$

and its complex conjugate. Here $F(\psi^*, \psi) = [(\frac{\omega_0}{2} + \frac{U}{2N}\psi^*\psi)U + 4g^2]/[2\sqrt{(\frac{\omega_0}{2} + \frac{U}{2N}\psi^*\psi)^2 + 4\frac{g^2}{N}\psi^*\psi}]$ and $G(\psi^*, \psi) = \sqrt{(\frac{\omega_0}{2} + \frac{U}{2N}\psi^*\psi)^2 + 4\frac{g^2}{N}\psi^*\psi}/2$. It is straightforward to find that Eq. (5) has a trivial solution $\psi = \psi^* = 0$ and the non-trivial solution $\psi(\tau) = \psi^*(\tau) = \pm\psi_0$, satisfying a nonlinear equation

$$\frac{2\omega\zeta}{(\omega_0 + U\bar{\psi}_0^2)U + 8g^2} = \tanh\left(\frac{\beta\zeta}{4}\right), \quad (6)$$

where $\zeta = \sqrt{(\omega_0 + U\bar{\psi}_0^2)^2 + 16g^2\bar{\psi}_0^2}$, and $\bar{\psi}_0^2 = \langle\psi^\dagger\psi\rangle/N$ is the scaled mean photon number. If $\psi = \psi^* = 0$, the system is in the NP with no collective excitation. However, the existence of non-trivial solutions shows that the macroscopic collective excitation for both atoms and photon occurs and thus the system locates at the SP. Finally, the stable conditions $\delta^2 S(\psi^*, \psi)/\delta\psi^2 > 0$ and $\delta^2 S(\psi^*, \psi)/\delta(\psi^*)^2 > 0$ determine the real solution of Eq. (5) and thus the phases of the BEC-cavity system at finite temperature.

When $U = 0$, Eq. (6) reduces to $\omega\sqrt{\omega_0^2 + 16g^2\bar{\psi}_0^2}/4g^2 = \tanh(\beta\sqrt{\omega_0^2 + 16g^2\bar{\psi}_0^2}/4)$, the result for the original Dicke model. In this case, only SP and NP exist, even at finite temperature [18]. When the temperature $T \rightarrow 0$, $\tanh(\beta\zeta/4) = 1$, and Eq. (6) becomes $2\omega\zeta/[(\omega_0 + U\bar{\psi}_0^2)U + 8g^2] = 1$, from which rich

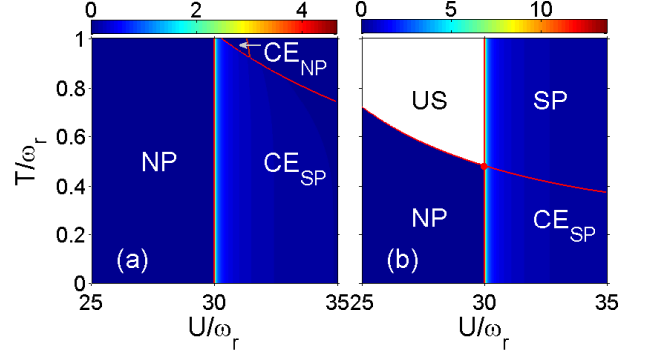


FIG. 4: (Color online) The scaled mean-photon number as the functions of the temperature T and the nonlinear interaction U for $g = 3.00 \omega_r$, $\Delta_c = 20.00\omega_r$ (a) and $g = 6.00 \omega_r$, $\Delta_c = 20.00\omega_r$ with $\omega_r = 2\pi \times 3.71$ kHz.

zero-temperature phase diagrams have been obtained [10–12]. With increasing temperature T , the function $\tanh(\beta\zeta/4)$ decreases. As a consequence, Eq. (6) can possess some new physical solutions of $\bar{\psi}_0$, leading to new phases at finite temperature.

We solve the nonlinear equation (5), together with stable conditions $\delta^2 S(\psi^*, \psi)/\delta\psi^2 > 0$ and $\delta^2 S(\psi^*, \psi)/\delta(\psi^*)^2 > 0$, for various different physical parameters and calculate the scaled mean photon number $\langle\psi^\dagger\psi\rangle/N = \bar{\psi}_0^2$. In Fig. 2, we plot the scaled mean-photon number $\bar{\psi}_0^2$ as the functions of the detuning Δ_c and the square of the collective coupling strength g for different temperatures (a) $T = 0.00$ nK, (b) $T = 0.13$ nK, (c) $T = 50.00$ nK, and (d) $T = 150.00$ nK. To compare with the experiment, we also choose the other parameters to be the same as those in Ref. [7], that is, the nonlinear interaction $U = 1.70 \times 10^3 \omega_r$, and the recoil frequency $\omega_r = 2\pi \times 3.71$ kHz. When $T = 0$, due to the existence of the strong nonlinear interaction, the CE_{NP} phase is found, apart from the NP and SP, as shown in Fig. 2 (a). In the CE_{NP} phase, $\bar{\psi}_0^2 \sim 0$, and almost no photon signature is generated. However, at the experimental operating temperature $T = 50.00$ nK, numerical solution of Eq. (5) shows that the finite-temperature critical point that separates the NP and SP agrees with that in the experiment. More interestingly, the CE_{NP} phase becomes the CE_{SP} phase, in which a weak photon signature (compared with the strong superradiant regime) emerges. We note that such weak photon signature for the CE_{SP} phase has already been observed in the experiment in the same parameter region, although it has not been explicitly pointed out. The transition temperature from the CE_{NP} phase at zero temperature to the CE_{SP} at 50 nK is $T_p = 0.13$ nK, as shown in Fig. 2(b). With increasing temperature T ($=150.00$ nK), the SP region becomes narrower due to the suppression of the superradiance by thermal fluctuations, as expected. However, a new phase diagram including the NP, SP, CE_{SP} , CE_{NP} and

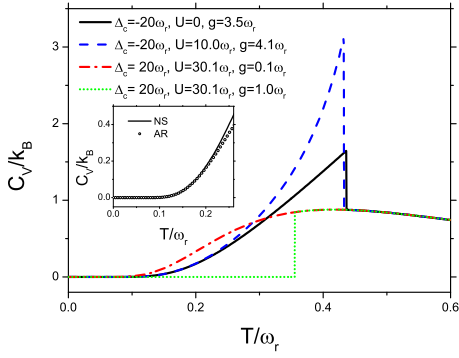


FIG. 5: (Color online) The specific heat C_V as a function of the temperature T for different nonlinear interaction U , detuning Δ_c and collective coupling strength g . Insert: comparison between the numerical simulation (NS) and the analytical result (AR) in Eq. (7) for the specific heat C_V .

US $(\delta^2 S(\psi^*, \psi) / \delta \psi^2 < 0$ and $\delta^2 S(\psi^*, \psi) / \delta (\psi^*)^2 < 0$) is observed, as shown in Fig. 2(d).

In a realistic experiment, the experimental parameters such as the detuning Δ_c , the collective coupling strength g , the nonlinear interaction U as well as the temperature T can be controlled independently. For example, Δ_c and g can be tuned by adjusting the frequency and power of the pump laser, respectively. The nonlinear interaction can be controlled by tuning both the frequency of the pump laser and the atom number. The temperature can be manipulated by controlling the evaporative cooling process for the BEC [23]. In Fig. 3, we plot the scaled mean-photon number $\bar{\psi}_0^2$ as a function of the temperature T and the collective coupling strength g . While in Fig. 4, we plot the scaled mean-photon number $\bar{\psi}_0^2$ as a function of the temperature T and the nonlinear interaction U . These figures show that rich finite-temperature phase diagrams exist in different parameter regions, induced by the competition among these parameters. Interestingly, we find a four-phase coexistence point in certain parameter region, as shown in Fig. 4(b). It should be emphasized again that these exotic phases arise from the interesting nonlinear interaction U and the finite temperature. If $U = 0$, only the SP and NP can be found [18].

These rich finite temperature phases and the phase transitions can be detected in experiments by measuring not only the mean photon number, but also some the thermodynamic quantities, such as the specific heat per atom $C_V = \frac{1}{Nk_B T^2} \frac{\partial^2}{\partial \beta^2} (\ln Z)$ in the BEC. In the NP, C_V can be obtained analytically, yielding $C_V^{\text{NP}} = \omega_0^2 [1 - \tanh^2(\beta\omega_0/4)] / (8k_B T^2)$ [2], which reaches the maximum at certain temperature (the red dashed-dotted line in Fig. 5). However, the explicit expression for C_V cannot be obtained in the SP. Nevertheless, in the region

$T \ll T_c$, we find

$$C_V^{\text{SP}} \simeq \frac{\zeta_0^2}{8k_B T^2} \text{sech}^2\left(\frac{\zeta_0}{4k_B T}\right) \quad (7)$$

where ζ_0 is the value of $\zeta = \sqrt{(\omega_0 + U\bar{\psi}_0^2)^2 + 16g^2\bar{\psi}_0^2}$ at $T = 0$. This expression shows that the specific heat C_V^{SP} in the SP increases rapidly (exponential increase at low temperature), which agrees well with the numerical results (see the insert of Fig. 5). At the critical temperature T_c , at which the system enters the NP, the specific heat C_V has a large jump. This step behavior is quite different from the behavior of the scaled mean photon number $\bar{\psi}_0^2$, which varies smoothly when crossing the critical point. Therefore the temperature-driven phase transition from SP to NP can be detected using the specific heat C_V . In addition, the behavior of specific heat C_V in the $\text{CE}_{\text{SP}}/\text{CE}_{\text{NP}}$ is similar to that in the SP/NP. Moreover, for the transition from the CE_{SP} to CE_{NP} phases, such a large jump of C_V still exists, as shown by the green dotted line of Fig. 5.

In summary, we develop a finite temperature theory for the Dicke phase transition of a BEC in an optical cavity. Our theory not only explains well the recent experimental observation of the phase transition from the SP to NP, but predicts rich new phases that either have been observed (but not pointed out) or may be observable by tuning experimental parameters. Our study is crucially important for understanding the Dicke phase transition in a realistic experimental system and may have significant application in quantum computation, quantum optics, etc.

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